GENERALIZATION OF THE EULER CASE OF MOTION OF A SOLID

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The motion of a free solid is Eulerian if it is not acted on by any external moments. It can be shown, however, that even in the presence of an external moment a solid can engage in motion which differs from the Eulerian only in the character of the time dependence of the angles, while the geometry of motion remains exactly the same as in the Euler case. In order for this to happen the external moment acting on the solid must maintain a constant direction in the inertial field and must be parallel to the kinetic moment vector **L** of the solid. The absolute value of this external moment **m** can be an arbitrary function of time.

Let $\omega_1, \omega_2, \omega_3$ be the projections of the angular velocity vector on the principal axes 1, 2, 3 of the solid whose principal moments of inertia are A_1, A_2, A_3. Assuming that $\alpha_1, \alpha_2, \alpha_3$ are the direction cosines of the vector **L** (and hence of the vector **m**) with respect to the axes 1, 2, 3 of the solid, we write the Eqs. of motion of the solid acted on by the moment **m** in the form

$$\begin{aligned} A_1\omega_1^{\cdot} + (A_3 - A_2)\,\omega_2\omega_3 &= \alpha_1 m \\ A_2\omega_2^{\cdot} + (A_1 - A_3)\,\omega_1\omega_3 &= \alpha_2 m \\ A_3\omega_3^{\cdot} + (A_2 - A_1)\,\omega_1\omega_2 &= \alpha_3 m \end{aligned} \tag{1}$$

By virtue of the above assumption concerning the external moment, the direction of the vector **L** in space is constant and can be taken as an axis of a stationary coordinate system. Then, making use of the self-evident relations

$$\alpha_1 L = A_1\omega_1, \qquad \alpha_2 L = A_2\omega_2, \qquad \alpha_3 L = A_3\omega_3 \tag{2}$$

we eliminate $\alpha_1, \alpha_2, \alpha_3$ from system (1),

$$\begin{aligned} A_1\omega_1^{\cdot} + (A_3 - A_2)\,\omega_2\omega_3 &= \frac{m}{L} A_1\omega_1 \\ A_2\omega_2^{\cdot} + (A_1 - A_3)\,\omega_1\omega_3 &= \frac{m}{L} A_2\omega_2 \\ A_3\omega_3^{\cdot} + (A_2 - A_1)\,\omega_1\omega_2 &= \frac{m}{L} A_3\omega_3 \end{aligned} \tag{3}$$

The resulting system enables us to construct two algebraic integrals which constitute a generalization of the integrals of the moment and the energy E in the Euler problem. In fact, multiplying Eqs. (3) by $A_1\omega_1, A_2\omega_2, A_3\omega_3$, respectively, and adding, we obtain the self-evident integral

$$L = \int m dt \tag{4}$$

Similarly, multiplying these Eqs. by $\omega_1, \omega_2, \omega_3$ and adding, we obtain the second integral

$$E / L^2 = \text{const} \tag{5}$$

Writing the resulting integrals in the form

$$\begin{aligned} A_1^2\omega_1^2 + A_2^2\omega_2^2 + A_3^2\omega_3^2 &= L^2 \\ A_1\omega_1^2 + A_2\omega_2^2 + A_3\omega_3^2 &= DL^2 \qquad (D = \text{const}) \end{aligned} \tag{6}$$

we note that they are completely identical in form to the corresponding integrals of the Euler problem, differing solely in the fact that L here is an explicit function of time given by relation (4). Moreover, we can reduce both these integrals and initial Eqs. (3) to a form entirely similar to the corresponding relations of the Euler case. This is accomplished by carrying out the substitution of variables

$$\omega_1 = \Omega_1 L, \quad \omega_2 = \Omega_2 L, \quad \omega_3 = \Omega_3 L, \quad \tau = \int L dt \tag{7}$$

where τ is the new independent variable.

Denoting differentiation with respect to τ with a prime, we obtain the following system for $\Omega_1, \Omega_2, \Omega_3$:

$$\begin{aligned} A_1\Omega_1' + (A_3 - A_2)\,\Omega_2\Omega_3 &= 0 \\ A_2\Omega_2' + (A_1 - A_3)\,\Omega_1\Omega_3 &= 0 \\ A_3\Omega_3' + (A_2 - A_1)\,\Omega_1\Omega_2 &= 0 \end{aligned} \tag{8}$$

Using the above to express $\Omega_1, \Omega_2, \Omega_3$ as elliptic functions of τ, we can easily use (7) to obtain $\omega_1, \omega_2, \omega_3$ as explicit functions of time. Clearly, the variability of L as compared with the Euler case results merely in a proportional variation of all the angular velocity vector components without altering the relationships among them. This enables us to represent sufficiently clearly the character of the phase point trajectory in the space $\omega_1, \omega_2, \omega_3$ for any law of variation of the external moment.

Turning now to the determination of the angles, we note that with allowance for the relations

$$\alpha_1 = \sin\vartheta\sin\varphi, \quad \alpha_2 = \sin\vartheta\cos\varphi, \quad \alpha_3 = \cos\vartheta \tag{9}$$

where ϑ and φ are the Euler angles of nutation and pure rotation, Expressions (2) imply that

$$\operatorname{tg}\varphi = \frac{A_1\Omega_1}{A_2\Omega_2}, \quad \cos\vartheta = A_3\Omega_3 \tag{10}$$

i.e. that the Euler angles ϑ and φ can be expressed in terms of τ exactly as in terms of t in the Euler case. As regards the precession angle ψ, eliminating the angles ϑ and φ from the relation

$$\psi^{\cdot} = \frac{\omega_1\sin\varphi + \omega_2\cos\varphi}{\sin\vartheta} \tag{11}$$

by means of (10) and converting to the new argument τ, we obtain

$$\psi' = \frac{A_1\Omega_1^2 + A_2\Omega_2^2}{A_1^2\Omega_1^2 + A_2^2\Omega_2^2} \tag{12}$$

This equation is quite identical in form to the corresponding equation in the Euler case. Thus, the angle ψ can be expressed in terms of τ exactly as in terms of t in the Euler case. All this means that the geometrical character of the motion of a solid acted on by an external moment of arbitrary magnitude parallel to the kinetic moment vector of the solid is completely identical to Eulerian motion. The only distinction lies in a different dependence of the angles on time and reduces to the replacement of the real time t by a quantity proportional to

$$\tau = \int\int m\,dt\,dt \tag{13}$$

Hence, the described state of motion can be regarded as a generalization of classical Eulerian motion subject to the same geometrical interpretation as in the Euler case.

In their famous mechanics courses Appell and MacMillan cite as an example a special case in which the equations of motion of a solid acted on by an external moment are reducible to Euler equations. This is the case where the moment acting on the solid is parallel and proportional to its kinetic moment. As is clear from the foregoing discussion, the latter requirement is quite superfluous.

Also noteworthy is the fact that if the moment acting on the solid is of constant absolute value, it can be shown that the process of acceleration (or deceleration) of the body is optimum in response (with certain limits as to the absolute value of the controlling moment).

Thus, the states of motion of a solid just considered are of definite practical interest.

Translated by A. Y.